Stochastic resonance in a mean-field model of cooperative behavior

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(Received 30 January 1995)

We study the long-time response of a stochastic system formed by very many interacting subsystems coupled by a mean-field interaction and subject to a time periodic external field. In the absence of a driving field, the system shows an order-disorder phase transition and its time evolution is well described by a Fokker-Planck equation which is nonlinear in the probability density. We carry out an analysis of the dynamics in the case of a weak driving field by means of a perturbation analysis (linear response theory). The response of the system is then given in terms of a generalized susceptibility. Its evaluation shows that the phenomenon of stochastic resonance, typical of driven bistable systems, is greatly enhanced by the dynamical feedback induced by mean-field coupling.

PACS number(s): 05.40.+j, 02.50.−r

I. INTRODUCTION

The subject of amplification of an external signal by the concerted action of the system dynamics and noise has been lately an active field of research [1]. In particular, a great deal of work has been devoted to the analysis of the response of a system characterized by a single degree of freedom, x, whose time evolution is governed by the Langevin equation (in dimensionless form),

$$\dot{x}(t) = x - x^3 + A\cos\Omega t + \eta(t), \tag{1}$$

where $A\cos\Omega t$ represents the effect of the external signal and $\eta(t)$ is a Gaussian noise with zero average and $\langle\eta(t)\eta(s)\rangle = 2D\delta(t-s)$. The corresponding linear Fokker-Planck equation (LFPE) for the probability density $P(x,t)$ is

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial x}\{(-x + x^3 - A\cos\Omega t)P\} + D\frac{\partial^2 P}{\partial x^2}. \tag{2}$$

The dynamics is that of a driven Brownian particle moving in a symmetric bistable potential in the limit of very large damping. The analysis of the problem can be carried out by making use of two important theorems: the H theorem, which ensures the existence of a uniquely determined long-time distribution function $P_\infty(x,t)$ and the Floquet theorem, which guarantees that $P_\infty(x,t)$ is periodic in time with the same period as the external force. In the limit of weak driving amplitudes and for driving frequencies smaller than the intrawell relaxation frequency, the phenomenon of stochastic resonance (SR) exists, so that the response of the system, measured by its long-time noise average, shows oscillations with amplitudes which can be much larger than the external amplitude and with a phase which is shifted with respect to that of the driving field. The nonmonotonic behaviors of the phase shift and the amplitude of the response with respect to the noise strength reflect the coherent use of the noise power by the system.

In the present work, we are interested in exploring the stochastic amplification of a sinusoidal signal in a model whose probability distribution obeys a nonlinear Fokker-Planck equation (NLFPE). The nonlinearity is brought about by the fact that the overall system consists of very many subunits with mean-field interactions among them [2]. The model will be presented in Sec. II. In the absence of external forcing, it can be shown that the system reaches equilibrium for long times, but the form of the equilibrium distribution function depends on the values of the parameters. Furthermore, in some regions of the parameter space, there are more than one stationary distribution and the system goes to one or the other depending upon its initial preparation. This bifurcation of the probability density is a consequence of the mean-field coupling and it is absent in the typical bistable LFP models described by Eq. (2).

When the mean-field model is acted upon by a weak time periodic force, it is possible to analyze its long-time behavior by means of perturbation theory around each of the stationary solutions, as shown in Sec. III. The response of the system is given in terms of a generalized susceptibility which can be constructed from the susceptibility of the model when the dynamical feedback due to the mean-field coupling is neglected. In Sec. IV, we calculate the generalized susceptibility. It shows a nonmonotonic behavior with the noise strength, pointing out that a resonant amplification of the input signal is feasible. The degree of amplification is larger than the one obtained with linear FP models. The results of the perturbation analysis are corroborated by the numerical solution of the Langevin equation, which is also carried out in Sec. IV.

II. THE MEAN-FIELD MODEL

Let us consider a set of N interacting subsystems, each one of them characterized by a single degree of freedom x_i $(i = 1, 2, \ldots, N)$, whose dynamics is governed by the Langevin equations

$$\dot{x_i} = x_i - x_i^3 + \frac{\vartheta}{N}\sum_{j=1}^{N}(x_j - x_i) + \eta_i(t), \tag{3}$$

where $\eta_i(t)$ is a Gaussian white noise with zero average and $\langle \eta_i(t)\eta_j(s)\rangle = 2D\delta_{ij}\delta(t-s)$, and ϑ is a parameter measuring the strength of the mean-field interaction between subsystems. This model was introduced by Kometani and Shimizu [3] to study the dynamics of muscle contraction. Later on, Desai and Zwanzig [2] and Dawson [4] gave a more complete statistical mechanical description of the model relating it to the Weiss-Ising model. They showed that, in the limit $N \to \infty$, all the subsystems have an identical evolution given by the nonlinear stochastic equation

$$\dot{x}(t) = (1-\vartheta)x(t) - x^3(t) + \vartheta\langle x(t)\rangle + \eta(t), \tag{4}$$

where $\langle x(t)\rangle = \lim_{N\to\infty} N^{-1}\sum x_i(t)$. According to the law of large numbers, $\langle x(t)\rangle$ does not fluctuate. It represents the time-dependent order parameter. The corresponding NLFPE is

$$\frac{\partial P(x,t)}{\partial t} = \frac{\partial}{\partial x}\{U'_{eff}[x,\langle x(t)\rangle,t]P(x,t)\} + D\frac{\partial^2 P(x,t)}{\partial x^2}, \tag{5}$$

where the prime indicates derivative with respect to x of the effective potential $U_{eff}[x,\langle x(t)\rangle t] = (\vartheta-1)\frac{x^2}{2} + \frac{x^4}{4} - \vartheta\langle x(t)\rangle x$, and $\langle x(t)\rangle$ satisfies

$$\langle x(t)\rangle = \int x\,P(x,t)\,dx. \tag{6}$$

Because of this condition, the FPE is a truly nonlinear evolution equation for the probability density. Depending on the values of the parameters D and ϑ, there might be more than one stable distribution. The functional form of the equilibrium solution of the NLPFE is obtained by setting $\frac{\partial P_{eq}}{\partial t} = 0$. This leads to

$$P_{eq}(x,\langle x\rangle_{eq}) = Z^{-1}\exp\left\{\frac{1}{D}U_{eff}(x,\langle x\rangle_{eq})\right\}, \tag{7}$$

where Z is the normalization factor. The equilibrium effective potential that shows up in Eq. (7) depends on $\langle x\rangle_{eq}$ which in turn has to comply with the condition

$$\langle x\rangle_{eq} = \int x\,P_{eq}(x,\langle x\rangle_{eq})\,dx. \tag{8}$$

The solution of this implicit equation yields the behavior of $\langle x\rangle_{eq}$ with the system parameters. It is clear that $\langle x\rangle_{eq} = 0$ is always a solution of Eqs. (7) and (8). On the other hand, the slope of the right-hand side of Eq. (8), considered as a function of $\langle x\rangle_{eq}$, changes from being less than 1 to greater than 1 at the origin, depending upon the values of the system parameters. Then, when the slope is equal to 1 there exists a critical line $1 = \frac{\vartheta}{D}[\langle x^2\rangle_{eq} - \langle x\rangle_{eq}^2]$ separating two distinct regions in the space of parameters (ϑ, D). For a given value of ϑ, there exists a $D = D_c$ such that for values greater than D_c the only stable equilibrium average is $\langle x\rangle_{eq} = 0$ and so, there is just one stable equilibrium probability density which has either one or two maxima depending on whether ϑ is larger or smaller than 1. For values of D smaller than the critical value, there are three solutions of Eq. (8). One of them $\langle x\rangle_{eq} = 0$ is unstable while the others two $\langle x\rangle_{eq} = \pm x_0$ are stable. Thus, in this region, there are two stable equilibrium distributions which are always single peaked. The system tends to one of them depending on the initial condition. We see then that this system shows a phase transition. For parameter values lying below the critical line, the system is said to be in a disordered phase ($\langle x\rangle_{eq} = 0$) while above the critical line the system is in an ordered phase ($\langle x\rangle_{eq} \neq 0$). At the critical line there is then a bifurcation of the probability density. In Fig. 1 we indicate the critical line in terms of the parameters θ and $|z| = |\theta - 1|(2D)^{-\frac{1}{2}}$. The equilibrium effective potential for each region is also sketched.

A few years ago, Shiino [5] was able to prove rigorously an important theorem which plays a role analogous to the H theorem of the LFP models. He introduced the quantity

$$H = \int P(x,t)\ln\frac{P(x,t)}{Q(x,t)}\,dx, \tag{9}$$

where the auxiliary function $Q(x,t)$ is defined as

$$Q(x,t) = \exp\left\{-\frac{1}{D}[U_{eff}(x,\langle x(t)\rangle,t)] - \frac{\vartheta}{2D}\langle x(t)\rangle^2\right\}. \tag{10}$$

Shiino proved that (i) $H(t)$ is bounded from below and (ii) $\frac{dH}{dt} \leq 0$, i.e., $H(t)$ is a monotonically decreasing function of time. Consequently, for long times, the system always relaxes to one of the equilibrium distributions with $\langle x\rangle_{eq}$ being one of the stable solutions of Eq. (8).

In this work we are interested in the dynamics of the system while being driven by an external time-dependent sinusoidal force. The dynamics of the relevant variable $x(t)$ is then described by the Langevin equation

$$\dot{x}(t) = (1-\vartheta)x(t) - x^3(t) + A\cos\Omega t + \vartheta\langle x(t)\rangle + \eta(t), \tag{11}$$

or, equivalently, by the NLFPE

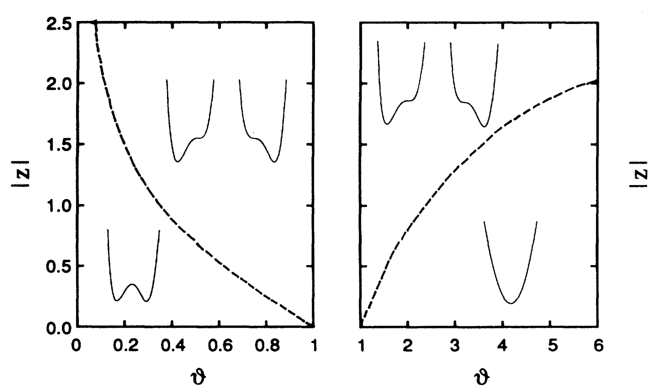

FIG. 1. Equilibrium phase diagram for the mean-field model. The dashed line is the critical line.

$$\frac{\partial P}{\partial t}=\frac{\partial}{\partial x}(\{U'_{eff}[x,\langle x(t)\rangle,t]-A\cos\Omega t\}P)+D\frac{\partial^2 P}{\partial x^2}. \quad (12)$$

We have not been able to find a quantity analogous to the H used by Shiino in the absence of time-dependent driving field. Also, by contrast with the LFP models, Eq. (12) is nonlinear and the Floquet theorem does not apply. Thus, in general, we cannot ensure that the long-time solutions of the NLFPE will be periodic. In the rest of the paper we will limit ourselves to the analysis of the system dynamics when the driving field has a small amplitude.

III. LINEAR RESPONSE THEORY (LRT)

Let us assume that the external field is so weak that its action can be treated perturbatively. Then, the probability density can be expanded as $P(x,t)=P^{(0)}(x,t)+P^{(1)}(x,t)$, where the first summand is the solution of the NLPFE in the absence of driving field. The noise average also splits up in two terms as $\langle x(t)\rangle=\langle x(t)\rangle^{(0)}+\langle x(t)\rangle^{(1)}$. From the discussion of the preceding section we know that $P^{(0)}(x,t)$ decays for long times to one of the equilibrium solutions. Thus, for long times, first order perturbation theory allows us to analyze the deviations of the probability density with respect to the equilibrium ones. It follows from Eq. (12) that

$$P^{(1)}(x,t)=-\int_{-\infty}^{t}d\tau\, e^{(t-\tau)\mathcal{D}^{(0)}(\langle x\rangle_{eq})}\times\left\{\vartheta\langle x(\tau)\rangle^{(1)}+A\cos\Omega t\right\}\frac{\partial P_{eq}}{\partial x}, \quad (13)$$

where $\langle x\rangle_{eq}$ is the stable equilibrium average for given values of the parameters D and ϑ of the unperturbed system. The linear FP operator $\mathcal{D}^{(0)}(\langle x\rangle_{eq})$ is given by

$$\mathcal{D}^{(0)}(\langle x\rangle_{eq})=\frac{\partial}{\partial x}[(\vartheta-1)x-x^3-\vartheta\langle x\rangle_{eq}]+D\frac{\partial^2}{\partial x^2}. \quad (14)$$

It is straightforward to show that $\mathcal{D}^0(\langle x\rangle_{eq})\,xP_{eq}=-U'_{eff}(x,\langle x\rangle_{eq})P_{eq}=D\frac{\partial P_{eq}}{\partial x}$.

Then it follows that

$$\begin{aligned}\langle x(t)\rangle^{(1)}&=\int dx\,x\,P^{(1)}(x,t)\\&=-\frac{1}{D}\int_{-\infty}^{t}d\tau\int dx\,[\vartheta\langle x(\tau)\rangle^{(1)}+A\cos\Omega\tau]\,x\,e^{(t-\tau)\mathcal{D}^{(0)}(\langle x\rangle_{eq})}\mathcal{D}^{(0)}(\langle x\rangle_{eq})\,xP_{eq}\\&=\int_{-\infty}^{t}d\tau[\vartheta\langle x(\tau)\rangle^{(1)}+A\cos\Omega\tau]K(t-\tau),\end{aligned} \quad (15)$$

where

$$K(t)=\begin{cases}-\frac{1}{D}\frac{d}{dt}\langle x(t)\,x\rangle_{eq}, & t>0\\ 0, & t<0.\end{cases} \quad (16)$$

The response function $K(t)$ is then related to the equilibrium time correlation function of the system in the absence of driving field. Notice that the stable equilibrium solution around which we are perturbing depends upon the values of the parameters and also on the initial condition when two stable distributions coexist. Using Fourier transform we get

$$\langle x(\omega)\rangle^{(1)}=R(\omega)\left\{\vartheta\langle x(\omega)\rangle^{(1)}+\frac{A}{2}[\delta(\omega-\Omega)+\delta(\omega+\Omega)]\right\}, \quad (17)$$

where

$$R(\omega)=\int_{0}^{\infty}dt\,K(t)\exp\{-i\omega t\}. \quad (18)$$

Thus, we have that the Fourier transform of the long-time deviation of the average value with respect to its corresponding equilibrium value is given by

$$\langle x(\omega)\rangle^{(1)}=A\chi(\omega)[\delta(\omega+\Omega)+\delta(\omega-\Omega)], \quad (19)$$

where

$$\chi(\omega)=\frac{R(\omega)}{1-\vartheta R(\omega)}. \quad (20)$$

Inverting Eq. (19) we see that, for long times, the response of the system shows oscillatory behavior around its corresponding equilibrium value as given by

$$\langle x(t)\rangle^{(1)}=A|\chi(\Omega)|\cos(\Omega t-\phi), \quad (21)$$

where we have written $\chi(\Omega)=|\chi(\Omega)|\exp(i\phi)$. In the next section, we will explicitly evaluate the generalized susceptibility $|\chi(\Omega)|$ to show that the response of the system presents stochastic resonant effects which are enhanced by the mean-field coupling with respect to the SR phenomena of the LFP models. The form of the response is similar to the one found by Dykman *et al.* [6] for a linear FP model in the absence of mean-field coupling ($\vartheta=0$), except that now the quantity measuring the degree of amplification is χ and not R. The static susceptibility $\chi(0)$ diverges at the critical line [5] $[1-\vartheta R(0)=0]$. Therefore, the relaxation time from any initial condition also diverges as the system is near the critical line. On the other hand, the amplitude of the oscillations of $\langle x(t)\rangle^{(1)}$ remains finite for nonzero frequencies. This behavior is valid within the limits of LRT, but one cannot guarantee that the same behavior will remain true for fields of any amplitude. We are presently exploring the response of the system to large driving fields, and the results seem to indicate a shift of the location of the critical line with respect to its zero-field value.

IV. ANALYSIS OF THE RESPONSE IN THE REGION OF EQUILIBRIUM BISTABLE POTENTIALS

As it is well known, the usual SR phenomena show up for stochastic systems whose dynamics is such that there are two attractors with noise-induced transitions between them. Thus we will explore the response of the system for the region of parameter space where the equilibrium distribution in the absence of external field is bimodal and the equilibrium effective potential is a symmetric double well: $\vartheta < 1$ and $|z| < |z_c|$ (see Fig. 1).

It is clear from the fluctuation-dissipation theorem that the response function $K(t)$ of a LFP model can be obtained from the knowledge of the equilibrium time correlation function of an undriven system. For linear FP models with bistable potentials several analytical approximations for the correlation function have been considered in the literature [7,8], which reflect the two relevant dynamics for the relaxation process: intrawell and interwell motions. It is then found that the susceptibility R shows a nonmonotonic behavior with the noise intensity so that the amplification of a weak input reaches its maximum at a value of the noise D_k for which the external frequency roughly matches twice the Kramers frequency of noise induced interwell jumps. For $D \ll D_k$ or $D \gg D_k$ the degree of amplification is small.

As pointed out in Sec. III, for the present mean-field model, the quantity related to the degree of amplification is χ given by Eq. (20). It is then clear that if $R(\omega)$ for a corresponding linear FP model is known from some experimental measurement or some numerical calculation, the generalized susceptibility $\chi(\omega)$ for the mean-field model can readily be evaluated. In a previous work [9], the average response of a linear model with a bistable potential was obtained by numerically solving the FPE. The method used was not restricted to very small values of the noise strength. The procedure of Ref. [9] provides a reliable way to obtain the complex susceptibility R for a LFP model with a bistable unperturbed potential $U_{eff}(x) = (\vartheta-1)\frac{x^2}{2}+\frac{x^4}{4}$ and for values of D so large that $|z| < |z_c|$. Once this function is known, the generalized susceptibility χ is obtained using Eq. (20).

In Fig. 2 we plot the behavior of $|\chi|$ with respect to $|z|$ for several values of ϑ, obtained using the procedure outlined above. The external frequency is kept fixed at $\Omega = 0.1$, while the amplitude of the driving field is taken to be $A = 0.032$ for $\vartheta = 0.1$. The height of the unperturbed barrier, given by $(1-\vartheta)^2/4$, decreases as ϑ increases. The amplitude of the external driving is adjusted accordingly, so that, for any value of ϑ, it is kept small enough for the linear response theory to remain valid. The curves for $\vartheta \neq 0$ end at $|z_c(\vartheta)|$, as we have

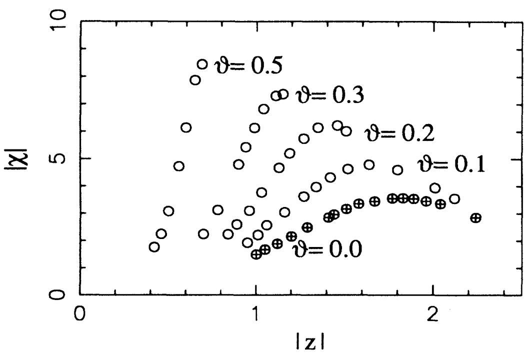

FIG. 2. Modulus of the dynamical susceptibility $|\chi|$ as a function of $|z|$ for $\Omega = 0.1$ and several values of the mean-field coupling parameter ϑ.

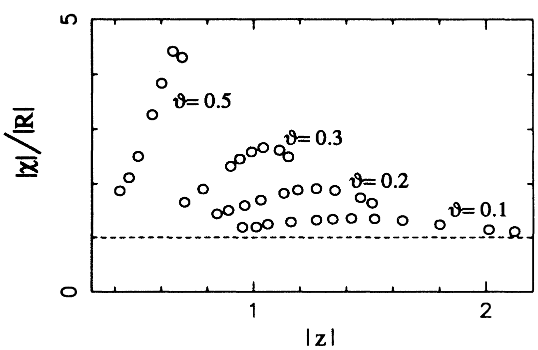

FIG. 3. Ratio between $|\chi|$ and $|R|$ as a function of $|z|$ for $\Omega = 0.1$ and several values of ϑ.

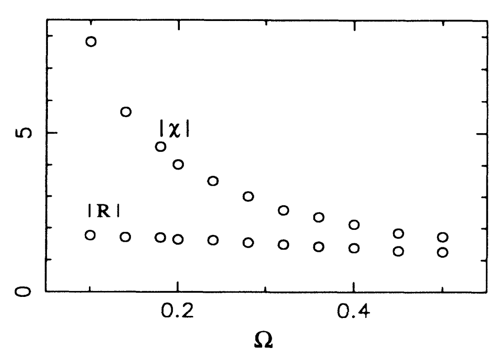

FIG. 4. Dependence of $|\chi|$ and $|R|$ with the driving frequency Ω for $\vartheta = 0.5$ and $|z| = 0.65$.

always kept our calculation restricted to the region where $|z| < |z_c(\vartheta)|$ and $\langle x\rangle_{eq} = 0$. We have not extended our calculations to $|z| > |z_c(\vartheta)|$ as in this region the unperturbed equilibrium effective potentials have a single well and no stochastic resonance amplification of a weak imput is expected there. Such restriction does not apply for the case $\vartheta = 0$ corresponding to the linear model.

It is clear that as the strength of the mean-field coupling is increased, the degree of amplification also increases. For $\vartheta = 0.1$, the point of maximun amplification takes place at $|z| \approx 1.7$ ($D \approx 0.14$), which is less than $|z_c(0.1)|$. If we use the Kramers formula to evaluate twice the hopping rate of a noise-induced transition in a bistable potential at this noise strength, we get

$$2\omega_{\rm Kr} = \frac{\sqrt{2}}{\pi}(1-\vartheta)\exp\left\{-\frac{z^2}{2}\right\} \approx 0.095, \qquad (22)$$

which roughly matches the driving frequency. This indicates that for small values of ϑ, the resonance mechanism is basically the same as in the case of linear FP models. As ϑ is increased, the peak in $|\chi|$ appears at decreasing values of $|z|$ and the corresponding noise strengths are too large for Kramers formula to apply. Therefore, the resonance mechanism for large values of ϑ is influenced by the mean-field coupling that has a dynamical effect, so that the average $\vartheta\langle x(t)\rangle$ acts as a feedback in the dynamics cooperating with the noise and the driving field to give rise to the large amplification. This is corroborated by the plots in Fig. 3, showing the ratio $|\chi|/|R|$ vs $|z|$ for several values of ϑ. The curves indicate that the generalized susceptibility $|\chi|$ is always greater than the susceptibility $|R|$ of a LFP system with potential $U(x) = (\vartheta - 1)\frac{x^2}{2} + \frac{x^4}{4}$.

We have also studied the influence of the driving frequency on the phenomenon of signal amplification. For driven linear FP models, $|R|$ decreases monotonously as Ω increases [8]. In Fig. 4 we plot the behavior of $|R|$ vs the driving frequency for the parameters $\vartheta = 0.5$ and $|z| = 0.65$. The decrease of $|R|$ as Ω increases is consistent with the behavior in linear models. This is because R is actually related to the response of a linear bistable model with a barrier height which depends on ϑ. We also show in Fig. 4 the behavior of $|\chi(\Omega)|$ for the same values of D and ϑ. Although the degree of amplification in a mean-field model also decreases as Ω increases, it is clear that the variation of $|\chi|$ with the frequency is much faster than that of $|R|$. This feature also reflects the influence of the dynamical effect of the mean-field coupling.

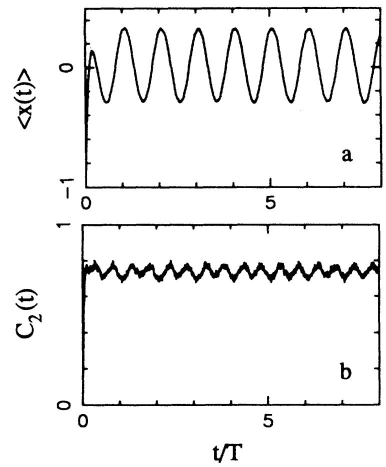

FIG. 5. Temporal evolution of the first two cumulants for $|z| = 1.16$, $\vartheta = 0.1$ and driving parameters $A = 0.1$ and $\Omega = 0.1$.

The response of the system to a driving field can also be analyzed by means of the Langevin equation. We have solved numerically Eq. (11) by generating a sufficiently large number of stochastic trajectories (5000 in most cases) and averaging over them. At each time step, the noise average $\langle x(t)\rangle$ is evaluated and its value is used as an input for all the noise realizations in the next time step. This technique was previously used by us for the same model in the absence of driving field and details can be found in Ref. [10]. In Fig. 5 we show the time behavior of the first two cumulants $\langle x(t)\rangle$ and $C_2(t) = \langle x^2(t)\rangle - \langle x(t)\rangle^2$ for $|z| = 1.16, \vartheta = 0.1$ and driving field parameters $A = 0.1$ and $\Omega = 0.1$. It is observed that, after a short transient, the average response $\langle x(t)\rangle$ shows an oscillation with an amplitude larger than the driving amplitude and a frequency equal to the driving frequency. The second cumulant also shows an oscillatory behavior around a steady value which is fairly large as it is characteristic of the cumulant of a bimodal distribution function. Similar behaviors are obtained with simulations for different values of $|z|$. The degree of amplification depends on $|z|$, so that, the resonance curve for $|\chi|$ obtained with the simulation of the Langevin equation matches the one shown in Fig. 2 using LRT.

In conclusion, we have seen that the dynamical feedback due to the mean-field coupling among very many interacting bistable systems driven by a weak time periodic field enhances the stochastic resonant effect with respect to the one obtained in a single bistable system.

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